Varianta 88

Subiectul I

a) 1. b) $\frac{4\sqrt{14}}{7}$. c) a = 0. d) $L\vec{M}(1,1,1)$, $M\vec{N}(1,1,1) \Rightarrow$ L, M, N coliniare. e) $\frac{41\sqrt{2}}{58}$.

f) a = -1. b = 0.

Subiectul II

1. a) 512. b) $\frac{1}{5}$. c) $g(5)=1$. d) x = 1. e) -1.

2. a) $f'(x)=2+\cos x+\sin x, \forall x\in \mathbf{R}$. b) $\int_0^{\frac{\pi}{2}} f(x)dx=\left(x^2-\cos x-\sin x\right)\Big|_0^{\frac{\pi}{2}}=\frac{\pi^2}{4}$.

c) $f'(x)=2+\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\geq 2-\sqrt{2}>0, \forall x\in \mathbf{R} \Rightarrow$ f este strict crescatoare pe $\mathbf{R}$.

d) $f'\left(\frac{\pi}{2}\right)=3$. e) $\frac{1}{16}\ln\left(4x^4+5\right)\Big|_0^1=\frac{1}{16}\ln\frac{9}{5}$.

Subiectul III

a) $O_2^2=O_2$, deci $O_2\in$ N, $A^2=O_2$, deci $A\in$ N.

b) $I_2^2=I_2\neq O_2$, deci $I_2\notin$ N;

c) $B^2=O_2 \Leftrightarrow \hat{a}^2+\hat{b}\cdot\hat{c}=\hat{0}, \left(\hat{a}+\hat{d}\right)\cdot\hat{b}=\hat{0}, \left(\hat{a}+\hat{d}\right)\cdot\hat{c}=\hat{0}, \hat{d}^2+\hat{b}\cdot\hat{c}=\hat{0} \Leftrightarrow \hat{a}^2+\hat{b}\cdot\hat{c}=\hat{0}$.

$\left(\hat{a}+\hat{d}\right)\cdot\hat{b}=\hat{0}; \left(\hat{a}+\hat{d}\right)\cdot\hat{c}=\hat{0}, \left(\hat{a}-\hat{d}\right)\cdot\left(\hat{a}+\hat{d}\right)=\hat{0}$. Daca $\hat{a}+\hat{d}\neq\hat{0}$ $\left(\hat{a}+\hat{d}=\hat{1}\right)$

atunci $\hat{a}=\hat{d}, \hat{b}=\hat{c}=\hat{0}$ si din $\hat{a}^2=\hat{0}\Rightarrow\hat{a}=\hat{0}$ deci $A=O_2$. Daca $\hat{a}+\hat{d}=\hat{0}$ ramane relatia $\hat{a}^2+\hat{b}\cdot\hat{c}=\hat{0}$ si cum $\hat{a}^2=\hat{a}\cdot\left(-\hat{d}\right)\Rightarrow\hat{a}\cdot\hat{d}-\hat{b}\cdot\hat{c}=\hat{0}$.

d) $C=\begin{pmatrix}\hat{1} & \hat{1}\\ \hat{0} & \hat{0}\end{pmatrix}$.

e) $2^4=16$.

f) $\hat{a}\in\{\hat{0},\hat{1}\}, \hat{d}=-\hat{a}\in\{\hat{1},\hat{0}\}$. Daca $\hat{a}=\hat{1}\Rightarrow A=\begin{pmatrix}\hat{1} & \hat{1}\\ \hat{1} & \hat{1}\end{pmatrix}$.

Daca $\hat{a}=\hat{0}\Rightarrow A_1=\begin{pmatrix}\hat{0} & \hat{0}\\ \hat{0} & \hat{0}\end{pmatrix}, A_2=\begin{pmatrix}\hat{0} & \hat{0}\\ \hat{1} & \hat{0}\end{pmatrix}, A_3=\begin{pmatrix}\hat{0} & \hat{1}\\ \hat{0} & \hat{0}\end{pmatrix}$. Deci N contine 4 elemente.

g) Matricea $D=\begin{pmatrix}\hat{1} & \hat{0}\\ \hat{0} & \hat{0}\end{pmatrix}$ are urma $\hat{a}+\hat{d}=\hat{1}$ si toate matricele din M au urma $\hat{0}$. Deci nu poate fi scrisa ca suma de elemente din M.

Subiectul IV

a) $f(0)=1$. $F(0)=0$.

b) f continua, deci admite primitive, si fie G o primitive a ei. Atunci

$F(x) = G(t)\Big|_0^x = G(x) - G(0)$, si din G derivabila avem G continua, deci

$G - G(0) = F$ derivabila si $F'(x) = G'(x) = f(x), \forall x \in \mathbf{R}$.

c) $F''(x) = f'(x) = 2xe^{x^2}$, $\forall x \in \mathbf{R}$, deci

$F''(x) < 0, \forall x \in (-\infty, 0)$, de unde F concava pe $(-\infty, 0\,]$ si

$F''(x) > 0, \forall x \in (0, \infty)$, de unde F convexa pe $[0, \infty)$.

d) $f(-x) = e^{(-x)^2} = e^{x^2} = f(x), \forall x \in \mathbf{R}$. $F(-x) = \int_0^{-x} f(t)dt$, notam y = -t, dy = -dt, iar pentru t = 0, y = 0 si pentru t = -x, y = x. Atunci

$F(-x) = -\int_0^x f(y)dy = -\int_0^x f(y)dy = -F(x), \forall x \in \mathbf{R}$.

e) Pentru x>1 $\int_0^x e^{t^2} dt > \int_1^x e^{t^2} dt > \int_1^x e^t dt = e^x - e$ si $\lim_{x\to\infty}(e^x - e) = \infty$, folosind criteriul majorarii, avem $\lim_{x\to\infty} F(x) = \infty$.

f) F continua si din d), e), avem $\lim_{x\to-\infty} F(x) = -\infty$, deci ImF = $\mathbf{R}$, coincide cu codomeniul si deci functia este surjectiva. Din $F'(x) = f(x) > 0, \forall x \in R$, obtinem ca F este strict crescatoare pe **R**, deci F este injective. In consecinta F este bijectiva.

g) Fie $t = g(x)\left(t = F^{-1}(x)\right)$, deci x = F(t), de unde $dx = F'(t)dt$. Pentru x = 0 din $F(t) = 0$ obtinem t = 0, iar pentru x = 1 $t = g(1) = a$:

$$\int_0^1 g(x)dx = \int_0^a tF'(t)dt = \int_0^a t \cdot f(t)dt = \int_0^a t \cdot e^{t^2} dt = \frac{1}{2}e^{t^2}\Big|_0^a = \frac{1}{2}\left(e^{t^2} - 1\right) = \frac{1}{2}(f(a) - 1)..$$